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# Waveform Design for MIMO Radar Using an Alternating Projection Approach

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***Abstract*— Revisiting an earlier examined multiple-input multiple-output (MIMO) radar waveform design problem which optimizes both minimum mean-square error estimation (MMSE) and mutual information (MI), we formulate a new waveform design problem and provide some further results in this paper, which complements the previous study. More specifically, we present an iterative optimization algorithm based on the alternating projection method, to determine waveform solutions that can simultaneously satisfy a structure constraint and optimize the design criteria. Numerical examples are provided, which illustrate the effectiveness of the proposed approach. In particular, we find that the waveform solutions obtained through our proposed algorithm can achieve very close and virtually indistinguishable performance from that predicted in the previous study.**

## I. Introduction

As multiple-input multiple-output (MIMO) radar has emerged as a new paradigm in the modern radar technology (see [1] for a review), a lot of efforts have been devoted to designing waveforms for MIMO radar to unleash its excellent performance advantages. Among the existing works is [2], which studies MIMO radar waveform design for the identification and classification of extended radar targets. It focuses on optimizing two criteria: maximization of the mutual information (MI) and minimization of the minimum mean-square error (MMSE), and requires the *a priori* knowledge of the extended target impulse response, e.g., the target's second-order statistics. With such knowledge and under some conditions, [2] indicates that under the same constraint that limits the total power, these two criteria lead to the same optimum solution, which employs water-filling and allocates the transmitted power in proportion to the quality of the particular mode in question. In a sequent paper [3], the minimax robust waveform design is investigated, which leads to different optimal waveforms under the two criteria above. In both [2] and [3], some fundamental quantity has been determined to characterize the essential part of the optimum waveform design, which can be deemed as the covariance matrix of the transmitted waveforms. However, unfortunately, the solutions obtained in [2] [3] are not in the ultimate form of the transmitted waveform matrix. Further, the solutions identified in [2] [3] require a certain condition embodied in, for example, Theorem 1 of [2], that the waveform matrix should have a specific Kronecker structure. Herein lies the purpose of this work: to find alternative formulations, without any restrictions, which provide directly the transmitted waveform matrix.

By explicitly taking into account the conditions required by Theorem 1 of [2], in this paper we reformulate the MIMO radar waveform design problem that was studied in [2]-[4]. We further present an iterative optimization algorithm to seek waveform solutions which both satisfy the structure condition and optimize the design criteria. Such algorithm, assuming the method of alternating projection as a basis, cyclically and alternatively computes the best approximations under different constraints. It also involves solving several matrix nearness problems, for which we resort to methods such as the Procrustean transformation and the Kronecker product approximation. We also provide some numerical examples, through which we find that the waveform solutions we obtain by applying the proposed algorithm, though are no longer in closed form, lead to performance which is very close to, and almost indistinguishable from that predicted by [2] [3].

The remainder of this paper is organized as follows. In Section II, we introduce the signal model and summarize the results about the optimal waveform design originated in [2]. In Section III, we reformulate the waveform design problem. In Section IV, we present an iterative optimization algorithm for the waveform design. In Section V, we provide numerical examples to validate the proposed waveform design. This paper is concluded in Section VI.

*Notation*: In this paper, we use the symbol $\|\cdot\|_F$ for the Frobenius norm of a matrix, $(\cdot)^+$ for the nonnegative part of a real-valued matrix, and $\otimes$ for the Kronecker product. $\Re\{\cdot\}$ and $\{\cdot\}^*$ denote the real part and the conjugate of a complex number, respectively.

## II. Signal Model and Optimal Waveform Design

We consider here a MIMO radar system with $P$ transmit and $Q$ receive antennas which may be distributed in space. We model the reflection from the signal sent from the $p$th transmit element and captured at the $q$th receive element using

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a finite impulse response (FIR) linear system with order $\nu$ in baseband, whose impulse response is $g^{(p,q)}(l),\ l=0,\cdots,\nu$. Then the received waveform at the $q$th receive element and discrete-time $k$ can be modeled as

$$y_q(k)=\sum_{p=1}^{P}\sum_{l=0}^{\nu} g^{(p,q)}(l)\, x_p(k-l)+n_q(k) \tag{1}$$

where $x_p(k)$ is the waveform transmitted from the $p$th transmit element and $n_q(k)$ is the additive complex Gaussian noise measured at the $q$th receive element. Let $L$ denote the length of the observed signal vector starting at an arbitrary discrete time $k$, and we assume that $L > PM$ (typically $L \gg PM$), where $M=\nu+1$. Denote $\mathbf{g}^{(p,q)} = [g^{(p,q)}(0),\ \cdots,\ g^{(p,q)}(\nu)]^T$, $\mathbf{y}_q=[y_q(k),\ \cdots,\ y_q(k+L-1)]^T$ and $\mathbf{n}_q=[n_q(k),\ \cdots,\ n_q(k+L-1)]^T$, then using the matrix-vector notation we can rewrite (1) as

$$\mathbf{y}_q=\sum_{p=1}^{P}\mathbf{X}_p\,\mathbf{g}^{(p,q)}+\mathbf{n}_q \tag{2}$$

where $\mathbf{X}_p$ is an $L\times M$ matrix which contains the waveforms transmitted from the $p$th transmit element. Defining $\bar{\mathbf{X}} = [\mathbf{X}_1,\ \cdots,\ \mathbf{X}_P]$ and $\bar{\mathbf{g}}_q=[(\mathbf{g}^{(1,q)})^T,\ \cdots,\ (\mathbf{g}^{(P,q)})^T]^T$ allows (2) to be expressed as

$$\mathbf{y}_q=\bar{\mathbf{X}}\,\bar{\mathbf{g}}_q+\mathbf{n}_q. \tag{3}$$

By stacking the received waveforms across all the $Q$ receive elements, we create $\bar{\mathbf{y}}=[{\mathbf{y}_1}^T,\ \cdots,\ \bar{\mathbf{y}}_Q^T]^T$. Defining $\boldsymbol{\mathcal{X}}=\mathbf{I}_Q\otimes\bar{\mathbf{X}}$, we finally obtain

$$\bar{\mathbf{y}}=\boldsymbol{\mathcal{X}}\bar{\mathbf{g}}+\bar{\mathbf{n}}, \tag{4}$$

where $\bar{\mathbf{g}}=[\bar{\mathbf{g}}_1^T,\ \cdots,\ \bar{\mathbf{g}}_Q^T]^T$ and $\bar{\mathbf{n}}=[{\mathbf{n}_1}^T,\ \cdots,\ {\mathbf{n}_Q}^T]^T$. We assume the target impulse response vector $\bar{\mathbf{g}}$ is a Gaussian random vector with zero mean and full rank covariance matrix $\boldsymbol{\Sigma}_{\bar{\mathbf{g}}}$. In particular, $\boldsymbol{\Sigma}_{\bar{\mathbf{g}}}$ can be diagonalized through $\boldsymbol{\Sigma}_{\bar{\mathbf{g}}} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H$, where $\boldsymbol{\Lambda}$ is a diagonal matrix with each diagonal entry given by a real and nonnegative eigenvalue of $\boldsymbol{\Sigma}_{\bar{\mathbf{g}}}$, and $\mathbf{U}$ is the corresponding unitary eigenvector matrix. We also assume components of the noise vector $\bar{\mathbf{n}}$ to be independently and identically distributed (i.i.d) and complex Gaussian, with mean 0 and variance $\sigma_n^2$.

The waveform design in [2] aims at optimizing two performance measures. One measure is the mean-square error resulting from estimating $\bar{\mathbf{g}}$ with a linear MMSE estimator, which can be expressed as

$$\xi=\mathrm{tr}\{(\sigma_n^{-2}\boldsymbol{\mathcal{X}}^H\boldsymbol{\mathcal{X}}+{\boldsymbol{\Sigma}_{\bar{\mathbf{g}}}}^{-1})^{-1}\}. \tag{5}$$

The other optimization metric is the conditional MI between $\bar{\mathbf{y}}$ and $\bar{\mathbf{g}}$ given the knowledge of $\bar{\mathbf{X}}$, which can be written as

$$\mathcal{I}=\log\left[\det(\sigma_n^{-2}\boldsymbol{\Sigma}_{\bar{\mathbf{g}}}\boldsymbol{\mathcal{X}}^H\boldsymbol{\mathcal{X}}+\mathbf{I}_{PQM})\right]. \tag{6}$$

Given the above two performance metrics, the optimal design problem can be formulated as to find a waveform matrix $\boldsymbol{\mathcal{X}}$ that either maximizes $\mathcal{I}$ in (6) or minimizes $\xi$ in (5), under a constraint on the total transmit power which is expressed as $\mathrm{tr}\{\boldsymbol{\mathcal{X}}^H\boldsymbol{\mathcal{X}}\}\le LQ\mathcal{P}_0$. It turns out these two criteria yield essentially the same optimum solution for $\boldsymbol{\mathcal{X}}^H\boldsymbol{\mathcal{X}}$, which is a fundamental quantity specifying the waveform design. Such equivalence between these two criteria is also true for an asymptotic formulation from [2] which requires only the knowledge of power spectral density of the target. Further, the optimum waveform matrix $\boldsymbol{\mathcal{X}}$ is given by [2, Theorem 1]

$$\boldsymbol{\mathcal{X}}=\boldsymbol{\Psi}\cdot[\left(\eta\mathbf{I}_{PQM}-\sigma_n^2\boldsymbol{\Lambda}^{-1}\right)^+]^{1/2}\cdot\mathbf{U}^H, \tag{7}$$

where $\boldsymbol{\Psi}$ is an $LQ\times PQM$ matrix with orthonormal columns (i.e., semi-unitary), and $\eta$ is a scalar constant which satisfies

$$\mathrm{tr}\{\left(\eta\mathbf{I}_{PQM}-\sigma_n^2\boldsymbol{\Lambda}^{-1}\right)^+\}=LQ\mathcal{P}_0. \tag{8}$$

For the asymptotic formulation, the optimal solution for $\boldsymbol{\mathcal{X}}$ will be identical to (7), and the only difference is that $\mathbf{U}$ is replaced by a unitary discrete Fourier transform (DFT) matrix [2, Theorem 2].

## III. Problem Formulation

Noteworthily, the condition given in (7) is only a necessary condition for $\boldsymbol{\mathcal{X}}$ to attain the optimality, but not a sufficient one. This is because there may exist some cases where a solution $\boldsymbol{\mathcal{X}}$ satisfies (7) but does not have the desired Kronecker structure. For these cases, it can be possible that the equivalence between the MI and the MMSE does not hold any more. Therefore, it is not only interesting but meaningful to seek a waveform solution or a better approximate for $\boldsymbol{\mathcal{X}}$ which simultaneously optimizes the performance measures of interest and satisfies the structure constraint. To be more specific, we would like to explicitly enforce the Kronecker structure in the waveform design. Towards this goal, we start from the optimality condition stated in (7), and formulate a new design problem as follows.

Let us define $\boldsymbol{\Pi}$ as a collection of $LQ\times PQM$ matrices $\boldsymbol{\mathcal{X}}$ that have the desired Kronecker structure. That is, for each $\boldsymbol{\mathcal{X}}\in\boldsymbol{\Pi}$, there exists a matrix $\bar{\mathbf{X}}\in\mathbb{C}^{L\times PM}$ which satisfies $\boldsymbol{\mathcal{X}}=\mathbf{I}_Q\otimes\bar{\mathbf{X}}$. We define a collection of semi-unitary matrices $\boldsymbol{\Psi}$ as

$$\boldsymbol{\Gamma}=\{\boldsymbol{\Psi}\in\mathbb{C}^{LQ\times PQM}:\boldsymbol{\Psi}^H\boldsymbol{\Psi}=\mathbf{I}_{PQM}\}.$$

$\boldsymbol{\Gamma}$ is essentially a Stiefel manifold [5, p. 301], which consists of all sets of $PQM$ orthonormal vectors in $\mathbb{C}^{LQ}$. We further define a $PQM\times PQM$ matrix

$$\boldsymbol{\Omega}\triangleq[\left(\eta\mathbf{I}_{PQM}-\sigma_n^2\boldsymbol{\Lambda}^{-1}\right)^+]^{1/2}\cdot\mathbf{U}^H. \tag{9}$$

The right hand side of (7) is then simplified as $\boldsymbol{\Psi}\boldsymbol{\Omega}$. Thus, the new waveform design problem can be stated as: to find a waveform matrix $\boldsymbol{\mathcal{X}}\in\boldsymbol{\Pi}$ (or equivalently a matrix $\bar{\mathbf{X}}$) that is minimally distant from $\{\boldsymbol{\Psi}\boldsymbol{\Omega}:\boldsymbol{\Psi}\in\boldsymbol{\Gamma}\}$ with respect to the Frobenius norm or in the sense of least squares. This problem, if cast mathematically, can be the following:

$$\begin{aligned}\textit{Problem 1}:\quad &\min_{\boldsymbol{\mathcal{X}},\boldsymbol{\Psi}} && \|\boldsymbol{\mathcal{X}}-\boldsymbol{\Psi}\boldsymbol{\Omega}\|_F^2\\ &\text{s.t.} && \boldsymbol{\mathcal{X}}\in\boldsymbol{\Pi},\\ & && \boldsymbol{\Psi}\in\boldsymbol{\Gamma}.\end{aligned}$$

## IV. An Iterative Optimization Algorithm

Note in *Problem 1*, the constraint sets are nonconvex. To solve this problem, we present an iterative optimization algorithm which features an alternating projection approach. Similar iterative algorithms have been suggested in [6] [7], and employed in [8] [9], for example. In fact, the method of alternating projection, firstly proposed by John von Neumann in 1933, has a long history and has found applications to many problems of different disciplines. It is an approach for finding the best approximation to any given point in a Hilbert space from the intersections of a finite number of subspaces, by alternatively computing the best approximations from the individual subspaces which make up the intersection [10].

When it comes to *Problem 1*, the alternating projection approach can start with a matrix in one set, e.g., $\boldsymbol{\mathcal{X}} \in \mathbf{\Pi}$, from which it computes a $\mathbf{\Psi} \in \mathbf{\Gamma}$ to minimize the Frobenius norm, i.e., the distance between the sets of $\{\mathbf{\Psi\Omega} : \mathbf{\Psi} \in \mathbf{\Gamma}\}$ and $\mathbf{\Pi}$. Next, the algorithm begins with the obtained $\mathbf{\Psi}$ to identify a $\boldsymbol{\mathcal{X}} \in \mathbf{\Pi}$ to minimize the distance. Then the algorithm repeats this alternating process indefinitely until a certain stopping rule is satisfied. This process is illustrated in Fig 1.

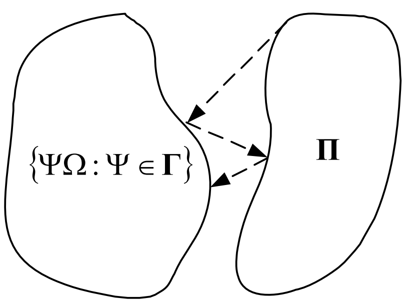



Fig. 1: An illustration of the alternating projection approach.

To this end, we summarize these procedures in Algorithm 1, where $\kappa$ denotes the maximum number of iterations. Alternatively, the algorithm can also be halted when additional iterations appear futile in further reducing the Frobenius norm. Note that the order of these two subproblems can be swapped. Also, at the initialization stage, the matrix, either $\boldsymbol{\mathcal{X}}$ or $\mathbf{\Psi}$, does not need to satisfy the constraint. It should be clear that Algorithm 1 never increases the Frobenius distance between successive iterations. In fact, it can be proved that Algorithm 1 converges to a global optimum in a weak sense. For detailed results with regard to its convergence properties, readers are referred to [7].

To apply Algorithm 1, we need to solve two minimization problems in step 3 and step 4. These two problems belong to the general class of matrix nearness problems (see [11] for a survey). We will discuss these two problems in detail shortly, but for notational simplicity we drop the iteration subscript. Notice that we will be able to obtain closed-form solutions for both minimization problems. This is one important appeal of Algorithm 1 as well as this design problem.

**Algorithm 1** Alternating Projection

**Input:** An initial matrix $\boldsymbol{\mathcal{X}}_0 \in \mathbb{C}^{LQ\times PQM}$

**Output:** $\widetilde{\boldsymbol{\mathcal{X}}}$ and $\widetilde{\mathbf{\Psi}}$

1: $i \leftarrow 0$;

2: **while** $i < \kappa$ **do**

3: find a matrix $\mathbf{\Psi}_i \in \mathbf{\Gamma}$ such that

$$\mathbf{\Psi}_i = \arg\min_{\mathbf{\Psi}\in\mathbf{\Gamma}} \|\boldsymbol{\mathcal{X}}_i - \mathbf{\Psi\Omega}\|_F^2;$$

4: find a matrix $\boldsymbol{\mathcal{X}}_{i+1} \in \mathbf{\Pi}$ such that

$$\boldsymbol{\mathcal{X}}_{i+1} = \arg\min_{\boldsymbol{\mathcal{X}}\in\mathbf{\Pi}} \|\boldsymbol{\mathcal{X}} - \mathbf{\Psi}_i\mathbf{\Omega}\|_F^2;$$

5: $i \leftarrow i+1$;

6: **end while**

7: $\widetilde{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{X}}_{i+1}$;

8: $\widetilde{\mathbf{\Psi}} = \mathbf{\Psi}_i$.

### *A. Procrustean Transformation*

In step 3 of Algorithm 1, we need to solve $\mathbf{\Psi} \in \mathbf{\Gamma}$ which minimizes $\|\boldsymbol{\mathcal{X}} - \mathbf{\Psi\Omega}\|_F^2$ provided $\boldsymbol{\mathcal{X}}$ is predetermined. Recall that $\{\mathbf{\Psi\Omega} : \mathbf{\Psi} \in \mathbf{\Gamma}\}$ consists of all possible rotations $\mathbf{\Psi\Omega}$ of $\mathbf{\Omega}$. Then this problem is equivalent to finding a rotation $\mathbf{\Psi\Omega}$ of $\mathbf{\Omega}$ which best approximates $\boldsymbol{\mathcal{X}}$ in the sense of least squares. In factor analysis, this type problem, particulary when $\mathbf{\Psi}$ is a unitary matrix, is usually referred to as the problem of finding a *Procrustean transformation* of $\boldsymbol{\mathcal{X}}$ [12, 7.4.8], or simply an *orthogonal Procrustes problem* [13, Sec. 12.4.1]. We summarize the solution to this problem in the following theorem.

*Theorem 1:* Let $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{LQ\times PQM}$ and $\mathbf{\Omega} \in \mathbb{C}^{PQM\times PQM}$ be given. Let $\mathbf{\Omega}\boldsymbol{\mathcal{X}}^H = \mathbf{VGW}^H$ be a singular value decomposition (SVD) of $\mathbf{\Omega}\boldsymbol{\mathcal{X}}^H$, where $\mathbf{V} \in \mathbb{C}^{PQM\times PQM}$ and $\mathbf{W} \in \mathbb{C}^{LQ\times PQM}$ are both unitary matrices, and $\mathbf{G} \in \mathbb{R}^{PQM\times PQM}$ is a diagonal matrix with nonnegative real numbers on the diagonal. Then the solution to the problem

$$\min_{\mathbf{\Psi}\in\mathbf{\Gamma}} \|\boldsymbol{\mathcal{X}} - \mathbf{\Psi\Omega}\|_F^2 \tag{10}$$

is $\mathbf{\Psi} = \mathbf{WV}^H$.

*Proof:* We can expand the least-squares expression in (10) as follows

$$\begin{aligned} & \|\boldsymbol{\mathcal{X}} - \mathbf{\Psi\Omega}\|_F^2 \\ = \;& \mathrm{tr}\{(\boldsymbol{\mathcal{X}} - \mathbf{\Psi\Omega})(\boldsymbol{\mathcal{X}} - \mathbf{\Psi\Omega})^H\} \\ = \;& \|\boldsymbol{\mathcal{X}}\|_F^2 - 2\Re\{\mathrm{tr}\{\mathbf{\Psi\Omega}\boldsymbol{\mathcal{X}}^H\}\} - \|\mathbf{\Omega}\|_F^2 \end{aligned}$$

Since $\mathrm{tr}\{\mathbf{\Psi\Omega}\boldsymbol{\mathcal{X}}^H\} = \mathrm{tr}\{\mathbf{\Omega}\boldsymbol{\mathcal{X}}^H\mathbf{\Psi}\}$, to mimize $\|\boldsymbol{\mathcal{X}} - \mathbf{\Psi\Omega}\|_F^2$, we need to find a matrix $\mathbf{\Psi}$ which solves the following problem

$$\max_{\mathbf{\Psi}\in\mathbf{\Gamma}} \Re\{\mathrm{tr}\{\mathbf{\Omega}\boldsymbol{\mathcal{X}}^H\mathbf{\Psi}\}\}.$$

Using $\mathbf{\Omega}\boldsymbol{\mathcal{X}}^H = \mathbf{VGW}^H$, we have

$$\Re\{\mathrm{tr}\{\mathbf{\Omega}\boldsymbol{\mathcal{X}}^H\mathbf{\Psi}\}\} = \Re\{\mathrm{tr}\{\mathbf{G}(\mathbf{W}^H\mathbf{\Psi}\mathbf{V})\}\},$$

which is maximized when $\mathbf{W}^H\mathbf{\Psi}\mathbf{V} = \mathbf{I}_{PQM}$ [12]. This leads to the solution $\mathbf{\Psi} = \mathbf{WV}^H$, and the value of the maximum

is $\mathrm{tr}\{\mathbf{G}\}$. This completes the proof. $\square$

### B. Kronecker Product Approximation

In step 4 of Algorithm 1, for a given matrix $\mathbf{\Psi}$, we need to find a waveform matrix $\boldsymbol{\mathcal{X}}$ which has the required Kronecker structure, i.e., $\boldsymbol{\mathcal{X}} = \mathbf{I}_Q \otimes \bar{\mathbf{X}}$. For notational convenience, we use $\bar{\mathbf{X}}$ explicitly instead of $\boldsymbol{\mathcal{X}}$ in this part, and further define $\mathbf{H} \triangleq \mathbf{\Psi\Omega}$. Thus, the problem can be rewritten as

$$\min_{\bar{\mathbf{X}}} \|\mathbf{H} - \mathbf{I}_Q \otimes \bar{\mathbf{X}}\|_F^2 \tag{11}$$

To solve the above problem, we can reformulate the Frobenius norm in (11) by appropriately breaking matrices $\bar{\mathbf{X}}$ and $\mathbf{H}$ into blocks, a method suggested in [14]. Thus, we obtain

$$\|\mathbf{H} - \mathbf{I}_Q \otimes \bar{\mathbf{X}}\|_F^2 = \sum_{k=1}^{L}\sum_{l=1}^{PM} \|\widehat{\mathbf{H}}(k,l) - \bar{\mathrm{X}}(k,l)\cdot \mathbf{I}_Q\|_F^2, \tag{12}$$

where we use the non-boldface letter $\bar{\mathrm{X}}(k,l)$ to denote the $(k,l)$-th element of matrix $\bar{\mathbf{X}}$. If using MatLab colon notation, $\widehat{\mathbf{H}}(k,l)$ can be expressed as

$$\widehat{\mathbf{H}}(k,l) = \mathbf{H}(k:L:LQ, l:PM:PQM).$$

In view of (12), we can minimize $\|\mathbf{H} - \mathbf{I}_Q \otimes \bar{\mathbf{X}}\|_F^2$ simply through a sequence of linear least squares optimizations. In such a process, each element of $\bar{\mathbf{X}}$ can be determined. We summarize this result in the following theorem.

*Theorem 2:* The matrix $\bar{\mathbf{X}} \in \mathbb{C}^{L\times PM}$ defined by

$$\bar{\mathrm{X}}(k,l) = \frac{1}{Q}\mathrm{tr}\{\widehat{\mathbf{H}}(k,l)\}, \quad 1 \le k \le L, 1 \le l \le PM \tag{13}$$

minimizes $\|\mathbf{H} - \mathbf{I}_Q \otimes \bar{\mathbf{X}}\|_F^2$.

*Proof:* In (12), we have expressed $\|\mathbf{H} - \mathbf{I}_Q \otimes \bar{\mathbf{X}}\|_F^2$ as a sum of a series of Frobenius norms, and only one term in the sum depends on $\bar{\mathrm{X}}(k,l)$ for a given $k$ and $l$. For each of these terms, corresponding to a given $k \in [1,L]$ and $q \in [1, PM]$, we have

$$\begin{aligned} &\|\widehat{\mathbf{H}}(k,l) - \bar{\mathrm{X}}(k,l)\cdot \mathbf{I}_Q\|_F^2 \\ =\ &\mathrm{tr}\{(\widehat{\mathbf{H}}(k,l) - \bar{\mathrm{X}}(k,l)\cdot \mathbf{I}_Q)(\widehat{\mathbf{H}}(k,l) - \bar{\mathrm{X}}(k,l)\cdot \mathbf{I}_Q)^H\} \\ =\ &\|\widehat{\mathbf{H}}(k,l)\|_F^2 - 2\Re\{\bar{\mathrm{X}}^*(k,l)\cdot \mathrm{tr}\{\widehat{\mathbf{H}}(k,l)\}\} + Q|\bar{\mathrm{X}}(k,l)|^2 \end{aligned}$$

which is minimized when

$$\bar{\mathrm{X}}(k,l) = \frac{1}{Q}\mathrm{tr}\{\widehat{\mathbf{H}}(k,l)\}.$$

Since $\bar{\mathrm{X}}(k,l)$ is specified, $\bar{\mathbf{X}}$ is also determined. $\square$

## V. Numerical Results

In this section, we present some numerical results to illustrate the performance of the waveforms obtained through the proposed algorithm. Let us firstly describe the performance benchmarks under the MMSE and MI measures. We denote the $(i,i)$-th element of the diagonal matrix $\mathbf{\Lambda}$ as $\Lambda(i,i)$. Thus, given the optimum solution (7)-(8), the resulting value of MMSE can be computed based on [2, Theorem 1], and is given by

$$\xi_{\min} = \sum_{i=1}^{PQM} \frac{\Lambda(i,i)}{(\Lambda(i,i)\sigma_n^{-2}\eta - 1)^+ + 1}. \tag{14}$$

Similarly, the resulting value of MI is [2]

$$\mathcal{I}_{\max} = \sum_{i=1}^{PQM} \left(\log(\sigma_n^{-2}\Lambda(i,i)\,\eta)\right)^+. \tag{15}$$

$\xi_{\min}$ in (14) and $\mathcal{I}_{\max}$ in (15) represent the theoretical optimal values of the objective functions (5) and (6), respectively. However, notice that when computing either $\xi_{\min}$ or $\mathcal{I}_{\max}$, the waveform matrix $\boldsymbol{\mathcal{X}}$ or $\bar{\mathbf{X}}$ is in fact not used. Also, recall that in the waveform design, the target's second-order statistics, i.e., $\mathbf{\Sigma}_{\bar{\mathbf{g}}} = \mathbf{U\Lambda U}^H$, are required *a priori*. Here we focus on solutions of the form discussed in [3], the asymptotic case where $\mathbf{\Sigma}_{\bar{\mathbf{g}}}$ is additionally assumed to have a Toeplitz structure, and can be approximated by its associated circulant matrix. In this case, we have

$$\mathrm{U}(k,l) = \frac{1}{\sqrt{PQM}}\exp\left\{\frac{-j2\pi(k-1)(l-1)}{PQM}\right\},$$

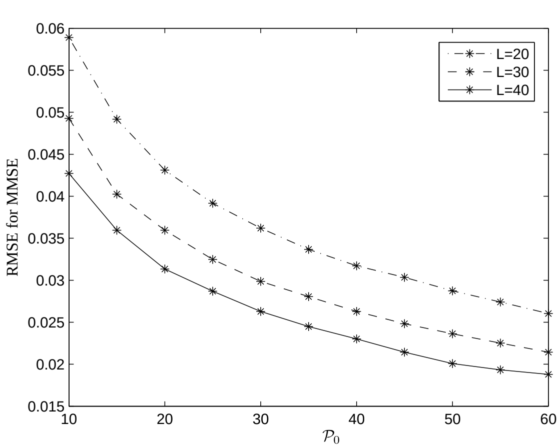
for $k,l \in [1, PQM]$. Thus the diagonal elements of $\mathbf{\Lambda}$ are the only knowledge we need about the target's statistics.

Our numerical experiments focus on a MIMO radar scenario of $P = Q = 2$ and $M = 5$, and we fix the noise power $\sigma_n^2$ at 2. For simplicity, but without loss of generality, for each single experiment we generate the diagonal elements of $\mathbf{\Lambda}$ independently from a uniform distribution over the range $[0, 10]$. We can then compute $\mathbf{\Omega}$ through (9) where a water-filling algorithm is employed. When implementing Algorithm 1, we start from a matrix $\boldsymbol{\mathcal{X}}_0$ which for convenience is initialized through $\boldsymbol{\mathcal{X}}_0 = \mathbf{I}_Q \otimes \bar{\mathbf{X}}_0$, with entries of $\bar{\mathbf{X}}_0$ given by i.i.d. complex Gaussian of zero mean and unity variance. We halt Algorithm 1 when the difference between subsequent values of $\|\boldsymbol{\mathcal{X}} - \mathbf{\Psi\Omega}\|_F$ is less than $10^{-5}$. After the waveform solution $\boldsymbol{\mathcal{X}}$ is obtained, we calculate the resulting values of MMSE and MI by simply plugging $\widetilde{\boldsymbol{\mathcal{X}}}$ into (5) and (6), respectively. We term these results as $\widetilde{\xi}_{\min}$ and $\widetilde{\mathcal{I}}_{\max}$.

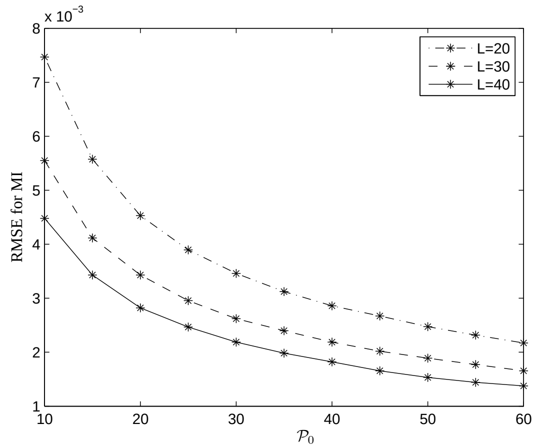
Our experiments indicate a close match between the theoretical optimal performance (denoted by $\xi_{\min}$ and $\mathcal{I}_{\max}$), and the one attained by using the resulting waveforms (denoted by $\widetilde{\xi}_{\min}$ and $\widetilde{\mathcal{I}}_{\max}$). However, due to limited space, we are unable to present all these results here. Instead, we consider another verification measure under both the MMSE and MI metrics, – the root mean-square error (RMSE). In this context, RMSE is defined as the square root of the mean of the squared relative errors over all the trials, i.e.,

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} e_i^2}, \tag{16}$$

where $N$ denotes the total number of independent trials and is set to 5000; $e_i$ denotes the *relative error* for the $i$th independent trial, and is defined to be the normalized difference between $\xi_{\min}$ ($\mathcal{I}_{\max}$) and $\widetilde{\xi}_{\min}$ ($\widetilde{\mathcal{I}}_{\max}$).




(a) RMSE for MMSE versus $L$ and $\mathcal{P}_0$




(b) RMSE for MI versus $L$ and $\mathcal{P}_0$

Fig. 2: RMSE under the MMSE and MI metrics and over $N = 5000$ independent trials

In Fig. 2a and Fig. 2b, we plot results under both the MMSE and MI metrics, respectively, for different values of $L$ and $\mathcal{P}_0$. It is observed that as either $L$ or $P_0$ increases, the performance difference between the theoretical optimal one and the one attained by using $\widetilde{\boldsymbol{\mathcal{X}}}$, in terms of RMSE, is reduced accordingly. Moreover, in view of the scale of those results depicted in Fig. 2a and Fig. 2b, we find that such performance difference, or the RMSE, seems almost negligible. In fact, the value of RMSE is also very small in each single experiment. This clearly demonstrates the effectiveness of our proposed algorithm, as well as the close-to-optimal feature of the obtained waveform solutions under the MMSE and MI performance measures. Additionally, it is worth making a quick note here, that the close-to-optimal performance attained by the resulting waveform solutions is also related to the denseness of solutions and the continuity of the performance measures.

## VI. Conclusion

In this paper, we revisited the MIMO radar waveform design problem originally investigated in [2] [3], which focuses on optimizing two criteria: maximization of MI and minimization of MMSE. We formulated a new waveform design problem, with an aim to seek waveform solutions which not only optimize the performance criteria of interest, but have a specific Kronecker structure. We presented an iterative optimization algorithm to find the waveform solutions. Such algorithm features the alternating projection method, and consists of operations like the Procrustean transformation and the Kronecker product approximation to provide closed-form solutions to some arising matrix nearness problems. The effectiveness of this proposed algorithm has been demonstrated through numerical examples. These numerical results also indicate that the waveform solutions obtained through the proposed algorithm can attain virtually indistinguishable performance when compared to that predicted in [2] [3]. As a result of this, it is reasonable to conclude the equivalence between the MI and MMSE criteria in this problem setup, which in fact corroborates the previous study in [2] [3].

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